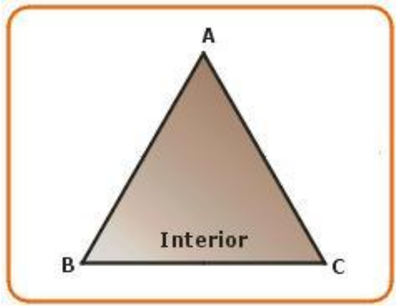
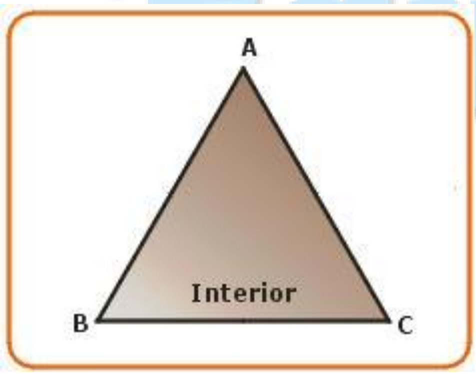
Areas of Parallelograms and Triangles

Triangle

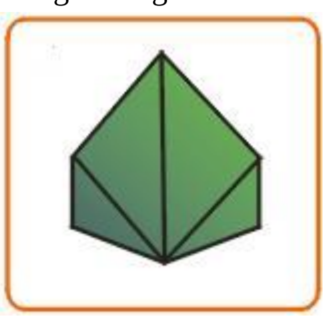
A plane figure bounded by three straight lines is called a triangle.
A triangle is the simplest polygon. It is a closed plane figure formed by three line segments. Hence a triangle has an area.



Consider the area of Δ ABC. It is the plane enclosed by the triangle. The **part** of the **plane enclosed** by the triangle is called the **interior** region of a **triangle.**
Hence the area a of a triangle is a **triangular region** formed by the union of a triangle and its interior.
Similarly the area of a rectangle and other polygons are polygonal regions.
A polygonal region is the union of a polygon and its interior.
Observe the diagrams of a rectangle and a hexagon.



A rectangle has two triangular regions. Hence the area of a rectangle is the union of two triangular regions.

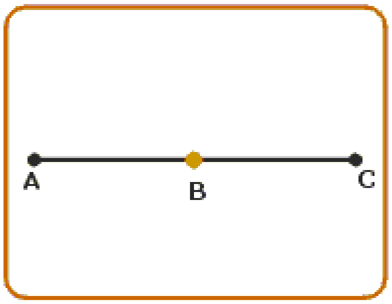


A polygonal region can be expressed as the union of a finite number of triangular regions.

Area of a Polygonal Region

The sides of a polygon are line segments and line segments have lengths. So it is natural to think that there may be some similar properties between the concept area of polygonal region and those of the length of a line segment.

Let us recall the concept of length.

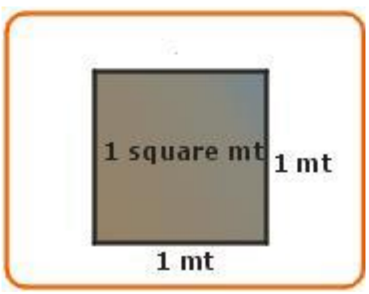


- To every line segment AB, there corresponds a unique positive real number called its **length**.

- If AB and CD are two congruent line segments then l(AB)= l(CD) where l refers to the length i.e., AB = CD then ⟺|(AB) =|(CD) or simply AB = CD
- If AB and CD are segments such that CD ⊆ AB,
- (CD is contained in AB) then | (CD) ≤ | (AB).
- If AB and BC are two line segments such that AC ∩ BC is a single point set, the | (AC) = | (AB) + | (BC).

You will find that the areas of regions behave in the same way as line segments.

Unit area



Every polygonal region has an area. There is a hundred square region of side (one) meter; called a square metre which is the unit area. The area of a polygonal region in square meters (sq - m or m^2) is a positive real number.

Notation of area

The area of a polygonal region R is denoted by ar (R). If ar(R) in square meters is x then we write ar (R) = xm^2.

Area axioms

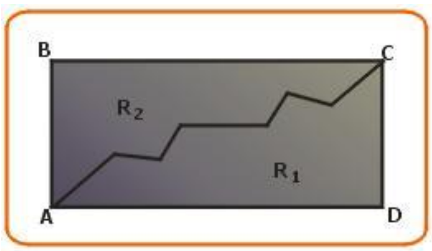
(a) Congruent area axiom

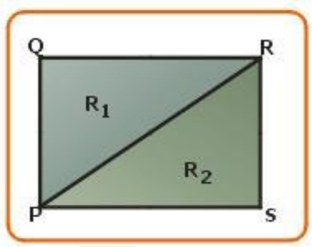
If $\Delta ABC \cong \Delta PQR$ then ar (Region of ΔABC) = ar (region ΔPQR)

(b) Area monotone axiom

If R_1 and R_2 are two polygonal regions such that $R_1 \subset R_2$ then ar $(R_1) \leq$ ar (R_2).

(c) Area addition axiom





If R_1 and R_2 are two polygonal regions whose intersection is either a finite number of line segments or single point and R = $R_1 + R_2$ then ar(R_0) = ar (R_1) + ar (R_2).

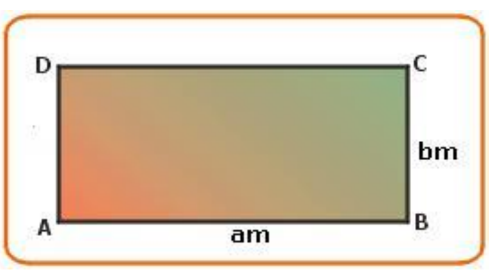
In figs (i) the region is divided into two regions R_1 and R_2. Note that $R_1 + R_2$ = R.

$\therefore$ ar (R) = ar (R_1) + ar (R_2)

Similarly in fig (ii),

ar (PQRS) = ar (R_1) + ar (R_2).

(d) Area of a rectangular region

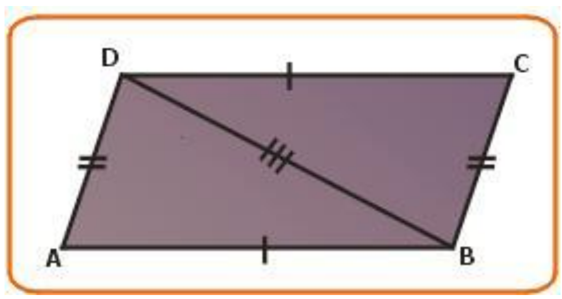


Given that AB = a metres and AD = b metres than ar (ABCD) = ab sq. m.

(Rect. area axiom)

Theorem 1

Statement:
Diagonals of a parallelogram divides it into two triangles of equal area.



Given:
ABCD is a parallelogram. AC is one of the diagonals of the parallelogram ABCD.

To prove:
$ar(\Delta ABD) = ar(\Delta DBC)$

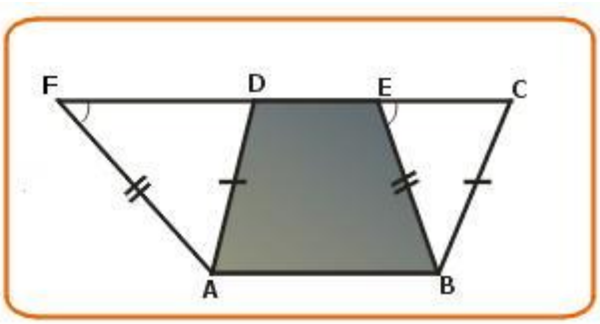
Proof:
In triangles ABD and DBC,
AB = DC (Opposite sides of parallelogram)
AD = BC (Opposite sides of parallelogram)
BD = BD (common side)

$\therefore \Delta ABD \cong \Delta CDB$ (sss congruency condition)

$\therefore ar(\Delta ABD) = ar(\Delta DBC)$

(area congruency axiom)

Theorem 2
Statement:
Parallelograms on the same base and between the same parallel lines are equal in area.



Given:

ABCD and ABEF are two parallelograms standing on the same base AB and between the same parallels AB and CF.

To prove:

$ar(||^m ABCD) = ar(||^m ABEF)$

Proof:

$ar(||^m ABCD) = ar(quad.\ ABED) + ar(\Delta EBC)$ ....(1) (area addition axiom)

$ar(||^m ABEF) = ar(quad.\ ABED) + ar(\Delta AFD)$ ....(2) (area addition axiom)

Now in triangles EBC and AFD,

AF = BE (opposite sides of a parallelogram)

AD = BC (opposite sides of a parallelogram)

$A\hat{F}D = B\hat{E}C$(AB || BE and FC is a transversal)

$\therefore$ $A\hat{F}D$ and $B\hat{E}C$ are corresponded angles

EF = AB = CD

EF - DE = CD - DE

i.e., FD = EC

$\Delta EBC \cong \Delta AFD$(SAS congruency condition)

$\therefore ar(\Delta EBC) = ar(\Delta AFD)$ .... (3)

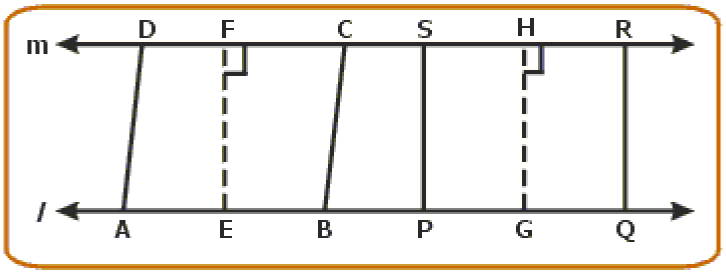
(area congruency condition)

From (1), (2) and (3),

$ar(||^m ABCD) = ar(||^m ABEF)$

Corollary

Statement:
Parallelograms on equal bases and between the same parallels are equal in area.



Given:

$||^m$ABCD and $||^m$ PQRS are between the same parallels l and m such that AB = PQ (equal bases).

To prove:
ar ($||^m$ ABCD) = ar ($||^m$ PQRS)

Construction:
Draw the altitude EF and GH.

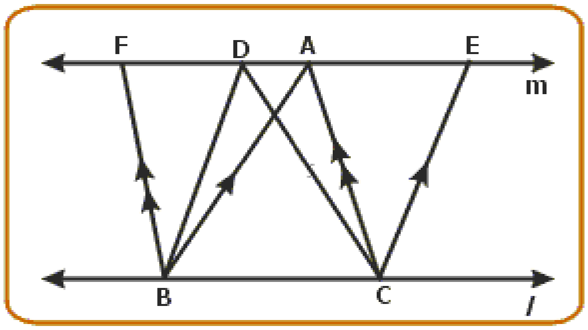
Proof:
l ||m (Given)
∴ EF = GH (perpendicular distance between the same parallels)
ar ($||^m$ ABCD) = AB x EF

(area of a $||^m$ = base x alr)
ar ($||^m$ PQRS) = PQ x GH

Since AB = GH (given)
and EF = GH (construction)
∴ ar ($||^m$ ABCD) = ar ($||^m$ PQRS)

Theorem 3

Statement:
Triangles on the same base and between the same parallels are equal in area.



Given:
Triangles ABC and DBC stand on the same BC and between the same parallels l and m.

To prove:
$ar(\Delta ABC) = ar(\Delta DBC)$

Construction:
CE || AB and BF || CA

Proof:
$||^m$ABCE and $||^m$ DCBF stand on the same base BC and between the same parallels l and m.

$\therefore\ ar(||^m\ ABCE) = ar(||^m\ DCBF)$ .... (1)

AC is a diagonal of $||^m$ABCE. It divides the parallelogram into two triangles of equal area.

$\therefore\ ar(\Delta ABC) = ar(\Delta ACE)$

or $ar(\Delta ABC) = \frac{1}{2}\ ar(||^m\ \Delta ABCE)$ .... (2)

Similarly we can prove that

$ar(\Delta DBC) = \frac{1}{2}\ ar(||^m\ DCBF)$ .... (3)

From (1), (2) and (3), we can write

$ar(\Delta ABC) = ar(\Delta DBC)$

Hence the theorem is proved.

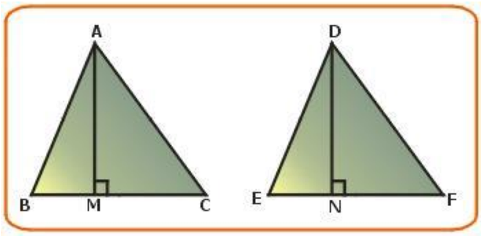
Theorem 4

Relation between the triangles of equal area and their corresponding altitudes

Recall that altitude is the perpendicular drawn from a vertex to its opposite side. Now let us draw two triangles of equal area and one side of one equal to the corresponding side of the other and find out the relationship between corresponding altitudes. Let us state the theorem.

Statement:

Triangles having equal areas and having one side of one of the triangle equal to one side of the other have their corresponding altitudes equal.



Given:

Two triangles ABC and DEF are such that

(i) ar (ΔABC) = ar (ΔDEF)

(ii) BC = EF

AM and DN are altitudes of triangle ABC and triangle DEF respectively.

To prove:

AM = DN

Proof:

In triangle ABC, AM is the altitude, BC is the base.

$$\therefore \quad ar(\Delta ABC) = \frac{1}{2} BC \times AM \quad .... (1)$$

In ΔDEF, DN is the altitude and EF is the base.

$$\therefore \quad ar(\Delta DEF) = \frac{1}{2} EF \times DN \quad .... (2)$$

$$\text{Since } ar(\Delta ABC) = ar(\Delta DEF)$$

$$\therefore \quad \frac{1}{2} BC \times AM = \frac{1}{2} EF \times DN \quad .... (3)$$

Also BC = EF (given)

∴ ½ AM = ½ DN

i.e., AM = DN.

Hence the theorem is proved.