

# BINOMIAL THEOREM & MATHEMATICAL INDUCTION

## BINOMIAL THEOREM

If  $a, b \in \mathbb{R}$  and  $n \in \mathbb{N}$ , then

$$(a + b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n a^0 b^n$$

**REMARKS :**

1. If the index of the binomial is  $n$  then the expansion contains  $n + 1$  terms.
2. In each term, the sum of indices of  $a$  and  $b$  is always  $n$ .
3. Coefficients of the terms in binomial expansion equidistant from both the ends are equal.
4.  $(a-b)^n = {}^nC_0 a^n b^0 - {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 - \dots + (-1)^n {}^nC_n a^0 b^n$ .

## GENERAL TERM AND MIDDLE TERMS IN EXPANSION OF $(A + B)^N$

$$t_{r+1} = {}^nC_r a^{n-r} b^r$$

$t_{r+1}$  is called a general term for all  $r \in \mathbb{N}$  and  $0 \leq r \leq n$ .  
Using this formula we can find any term of the expansion.

**MIDDLE TERM (S) :**

1. In  $(a + b)^n$  if  $n$  is even then the number of terms in the expansion is odd. Therefore there is only one

middle term and it is  $\left(\frac{n+2}{2}\right)^{\text{th}}$  term.

2. In  $(a + b)^n$ , if  $n$  is odd then the number of terms in the expansion is even. Therefore there are two middle terms and those are

$\left(\frac{n+1}{2}\right)^{\text{th}}$  and  $\left(\frac{n+3}{2}\right)^{\text{th}}$  terms.

## BINOMIAL THEOREM FOR ANY INDEX

If  $n$  is negative integer then  $n!$  is not defined. We state binomial theorem in another form.

$$(a+b)^n = a^n + \frac{n}{1!} a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2$$

$$+ \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!} a^{n-r} b^r + \dots$$

Here  $t_{r+1} = \frac{(n-1)(n-2)\dots(n-r+1)}{r!} a^{n-r} b^r$

**THEOREM:**

If  $n$  is any real number,  $a = 1$ ,  $b = x$  and  $|x| < 1$  then

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots$$

Here there are infinite number of terms in the expansion, The general term is given by

$$t_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)x^r}{r!}, r \geq 0$$

*Note...*

- (i) Expansion is valid only when  $-1 < x < 1$
- (ii)  ${}^nC_r$  can not be used because it is defined only for natural number, so  ${}^nC_r$  will be written as  $\frac{n(n-1)\dots(n-r+1)}{r!}$
- (iii) As the series never terminates, the number of terms in the series is infinite.
- (iv) General term of the series  $(1 + x)^{-n} = T_{r+1} \rightarrow (-1)^r \frac{1 + x}{1 - x}$  if  $|x| < 1$
- (v) General term of the series  $(1 - x)^{-n} \rightarrow T_{r+1} = \frac{(+1)(+2)\dots(+r-1)}{r!} x$
- (vi) If first term is not 1, then make it unity in the following way.  $(a + x)^n = a^n (1 + x/a)^n$  if  $\left|\frac{x}{a}\right| < 1$

REMARKS:

1. If  $|x| < 1$  and  $n$  is any real number, then

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

The general term is given by

$$t_{r+1} = \frac{(-1)^r n(n-1)(n-2)\dots(n-r+1)}{r!} x^r$$

2. If  $n$  is any real number and  $|b| < |a|$ , then

$$= (a+b)^n = \left[ a \left( 1 + \frac{b}{a} \right) \right]^n$$

$$= a^n \left( 1 + \frac{b}{a} \right)^n$$



While expanding  $(a+b)^n$  where  $n$  is a negative integer or a fraction, reduce the binomial to the form in which the first term is unity and the second term is numerically less than unity.

Particular expansion of the binomials for negative index,  $|x| < 1$

1.  $\frac{1}{1+x} = (1+x)^{-1}$   
 $= 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$

2.  $\frac{1}{1-x} = (1-x)^{-1}$   
 $= 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$

3.  $\frac{1}{(1+x)^2} = (1+x)^{-2}$   
 $= 1 - 2x + 3x^2 - 4x^3 + \dots$

4.  $\frac{1}{(1-x)^2} = (1-x)^{-2}$   
 $= 1 + 2x + 3x^2 + 4x^3 + \dots$

**BINOMIAL COEFFICIENTS**

The coefficients  ${}^n C_0, {}^n C_1, {}^n C_2, \dots, {}^n C_n$  in the expansion of  $(a+b)^n$  are called the binomial coefficients and denoted by  $C_0, C_1, C_2, \dots, C_n$  respectively

Now

$$(1+x)^n = {}^n C_0 x^0 + {}^n C_1 x^1 + {}^n C_2 x^2 + \dots + {}^n C_n x^n \quad \dots (i)$$

Put  $x = 1$ .

$$(1+1)^n = {}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n$$

$$\therefore 2^n = {}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n$$

$$\therefore {}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n = 2^n$$

$$\therefore C_0 + C_1 + C_2 + \dots + C_n = 2^n$$

$\therefore$  The sum of all binomial coefficients is  $2^n$ .

Put  $x = -1$ , in equation (i),

$$(1-1)^n = {}^n C_0 - {}^n C_1 + {}^n C_2 - \dots + (-1)^n {}^n C_n$$

$$\therefore 0 = {}^n C_0 - {}^n C_1 + {}^n C_2 - \dots + (-1)^n {}^n C_n$$

$$\therefore {}^n C_0 - {}^n C_1 + {}^n C_2 - {}^n C_3 + \dots + (-1)^n {}^n C_n = 0$$

$$\therefore {}^n C_0 + {}^n C_2 + {}^n C_4 + \dots = {}^n C_1 + {}^n C_3 + {}^n C_5 + \dots$$

$$\therefore C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots$$

$C_0, C_2, C_4, \dots$  are called as even coefficients

$C_1, C_3, C_5, \dots$  are called as odd coefficients

$$\text{Let } C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = k$$

$$\text{Now } C_0 + C_1 + C_2 + C_3 + \dots + C_n = 2^n$$

$$\therefore (C_0 + C_2 + C_4 + \dots) + (C_1 + C_3 + C_5 + \dots) = 2^n$$

$$\therefore k + k = 2^n$$

$$2k = 2^n$$

$$\therefore k = \frac{2^n}{2}$$

$$\therefore k = 2^{n-1}$$

$$\therefore C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$$

$$\therefore \text{The sum of even coefficients} = \text{The sum of odd coefficients} = 2^{n-1}$$

**Properties of Binomial Coefficient**

For the sake of convenience the coefficients

${}^n C_0, {}^n C_1, \dots, {}^n C_r, \dots, {}^n C_n$  are usually denoted by  $C_0, C_1, \dots, C_r, \dots, C_n$  respectively.

(i)  $C_0 + C_1 + C_2 + \dots + C_n = 2^n$

(ii)  $C_0 - C_1 + C_2 - \dots + (-1)^n C_n = 0$

(iii)  $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$ .

(iv)  ${}^n C_{r_1} = {}^n C_{r_2} \Rightarrow r_1 = r_2 \text{ OR } r_1 + r_2 = n$

(v)  ${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$

(vi)  $r {}^n C_r = n {}^{n-1} C_{r-1}$

**Some Important Results**

(i)  $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ ,

Putting  $x = 1$  and  $-1$ , we get

$C_0 + C_1 + C_2 + \dots + C_n = 2^n$  and

$C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n C_n = 0$

(ii) Differentiating  $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ ,

on both sides we have,  $n(1+x)^{n-1}$

$= C_1 + 2C_2x + 3C_3x^2 + \dots + nC_nx^{n-1}$  ....(1)

$x = 1$

$\Rightarrow n2^{n-1} = C_1 + 2C_2 + 3C_3 + \dots + nC_n$

$x = -1$

$\Rightarrow 0 = C_1 - 2C_2 + \dots + (-1)^{n-1} nC_n$ .

Differentiating (1) again and again we will have different results.

(iii) Integrating  $(1+x)^n$ , we have,

$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1}$

(where C is a constant)

Put  $x = 0$ , we get  $C = -\frac{1}{(n+1)}$

Therefore

$\frac{(1+x)^{n+1} - 1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \dots + \frac{C_nx^{n+1}}{n+1}$  ... (2)

Put  $x = 1$  in (2) we get

$\frac{2^{n+1} - 1}{n+1} = C_0 + \frac{C_1}{2} + \dots + \frac{C_n}{n+1}$

Put  $x = -1$  in (2) we get,

$\frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots$

**Illustration**

Find the coefficient of  $x^4$  in the expansion of

$\frac{1+x}{1-x}$  if  $|x| < 1$

**Sol.**  $\frac{1+x}{1-x} = (1+x)(1-x)^{-1}$

$= (1+x) \left[ 1 + \frac{(-1)}{1!}(-x) + \frac{(-1)(-1-1)}{2!}(-x)^2 \right.$

$\left. + \frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3 + \dots \text{to } \infty \right]$

$= (1+x)(1+x+x^2+x^3+x^4+\dots \text{to } \infty)$

$= [1+x+x^2+x^3+x^4+\dots \text{to } \infty] +$

$[x+x^2+x^3+x^4+\dots \text{to } \infty]$

$= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + \dots \text{to } \infty$

Hence coefficient of  $x^4 = 2$

**Illustration**

Find the square root of 99 correct to 4 places of decimal.

**Sol.**  $(99)^{1/2} = (100-1)^{1/2} \left[ 100 \left( 1 - \frac{1}{100} \right) \right]^{1/2}$

$= \left[ 100 \left( 1 - \frac{1}{100} \right) \right]^{1/2}$

$= (100)^{1/2} [1-0]^{1/2} = 10(1-0)^{1/2}$

$10 \left[ 1 + \frac{1}{2}(-0.01) + \frac{1}{2} \left( \frac{1}{2} - 1 \right) \frac{(-0.01)^2}{2!} + \dots \text{to } \infty \right]$

$= 10 [1 - 0.005 - 0.0000125 + \dots \text{to } \infty]$

$= 10(99.99875) = 9.999875 \approx 9.9999$

**Multinomial Expansion**

In the expansion of  $(x_1 + x_2 + \dots + x_n)^m$  where  $m, n \in \mathbb{N}$  and  $x_1, x_2, \dots, x_n$  are independent variables, we have

(i) Total number of terms  $= m^{n-1} C_{n-1}$

(ii) Coefficient of  $x_1^{r_1} x_2^{r_2} x_3^{r_3} \dots x_n^{r_n}$  (where  $r_1 + r_2 + \dots + r_n = m, r_i \in \mathbb{N} \cup \{0\}$ ) is  $\frac{m!}{r_1! r_2! \dots r_n!}$

(iii) Sum of all the coefficients is obtained by putting all the variables  $x_i$  equal to 1.

**Illustration**

Find the total number of terms in the expansion of  $(1+a+b)^{10}$  and coefficient of  $a^2b^3$ .

**Sol.** Total number of terms  $= 10+3-1 C_{3-1} = {}^{12}C_2 = 66$

Coefficient of  $a^2b^3 = \frac{10!}{2! \times 3! \times 5!} = 2520$